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Random functionals on $K\{M_p\}$ spaces

by

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Let V be a linear topological space, $(\Omega, \mathscr{A})$ a probability space, and $X(w, v)$, $w \in \Omega$, $v \in V$ a complex valued function on $\Omega \times V$ such that $X(\cdot, v)$ is a random variable for each $v \in V$. By analogy with Yaglom [7], X is called a *random field* or for $V = R$, a *stochastic process*, or in general, a *random functional.*

Applications frequently require additional properties relative to V, i. e. continuity, differentiability or joint properties such as mean-square continuity, stationarity, covariance stationarity or mean-square differentiability. Gel'fand [1], Ullrich [5], Yaglom [7] in particular have considered applications involving differentiability defined in terms of test function spaces, with Yaglom utilizing $V = S$, the slowly increasing functions and Ullrich, $V = \mathscr{D}$, the Schwartz space. In both of these latter instances, the random functional is considered as a mapping from Ω to S^* or $\mathscr{D}^*$, the dual spaces with an appropriate measurability condition, rather than as a function on $\Omega \times V$ as indicated above.

For both S and $\mathscr{D}$, representation theorems for random functionals have been obtained, [7], [6], which parallel those for elements of S^*, $\mathscr{D}^*$. As was observed by Yaglom and Ullrich, random functionals on these spaces are special instances of those of Gel'fand.

In this paper we will consider random functionals on $K\{M_p\}$ and on inductive limits of $K(M_p\}$ spaces. Since S and $\mathscr{D}$ can be obtained as special instances of $K\{M_p\}$ spaces by appropriate choice of the functions M_p, many of the results of Yaglom and Ullrich are obtained by those choices. A principal result is Theorem 2, which provides a representation for random functionals on this large class of spaces.

I. $K\{M_p\}$ **spaces.** We refer the reader to [2], p. 87, for the definition and pertinent properties of $K\{M_p\}$ spaces, also for the properties denoted (M), (N), (P). For inductive limits of $K\{M_p\}$ spaces see [2], p. 58, an inductive limit space being denoted by $\mathscr{K}$.

Throughout the rest of this paper we will abbreviate $K\{M_p\}$ by K and $K\{M_p^j\}$ by K^j, with $\{M_p\}$ or $\{M_p^j\}$ fixed, given sequences, unless noted otherwise.

LEMMA 1. *Let f be essentially bounded on R^n and suppose there exists p such that $M_p f$ induces an element of K^*. If $\{M_p\}$ satisfies conditions* (M) *and* (N) *and if polynomials are multipliers on K, then*

$$\int^{x_i} M_p f = \int_0^{x_i} \ldots M_p(t) f(t)\, dt_i$$

induces an element of K^* and $D_i\left(\int^{x_i} M_p f\right) = M_p f$, where $D_i = \partial/\partial x_i$ is differentiation in the sense of elements of K^*.

Since the proof for R^n is analogous to the case for $n = 1$, we will write the proof in that form: By (N), there exists $p' > p$ such that $M_p/M_{p'} = M_{pp'}$ is in $L^1(R)$. Also from (M)

$$*\left|\frac{1}{M_{p'}(s)} \int_0^s M_p(t) f(t)\, dt\right| \leqslant \frac{M_p(s)}{C_p M_{p'}(s)} \left|\int_0^s \right| f(t)|\, dt \leqslant \frac{M_{pp'}(s)}{C_p} \|f\|_\infty |s|.$$

Since the polynomials are multipliers on K and $M_{pp'}$ is in $L^1(R)$ it follows that $\int_0^x M_p f$ induces an element of K^* (see [2], p. 83). Let T be the element induced by $\int_0^x M_p f$; then

$$(DT)(\varphi) = -T(D\varphi) = -\int_R \left[\int_0^x M_p(t) f(t)\, dt\right] \varphi'(x)\, dx = \int_R M_p(x) f(x)\, \varphi(x)\, dx.$$

Since by (N), φ vanishes at ∞, $D\int_0^x M_p f = M_p f$.

II. Random linear functionals on $K\{M_p\}$ spaces. We follow Ullrich [5] in defining these. Suppose given a fixed probability space $(\Omega, \mathscr{A}, \mu)$. For S a non-empty set and $\mathscr{S}$ a σ-algebra over S and $E \subset S$, denote by $E \cap \mathscr{S}$, the minimal σ-algebra generated by sets of the form $E \cap F$, $F \in \mathscr{S}$. Now let Φ be any test function space and Φ^* the dual space; and Φ_r^* the σ-algebra over Φ^* generated by sets of the form

$$\{T/\mathrm{Re}(T(\varphi)) < C_1,\ \mathrm{Im}(T(\varphi)) < C_2\}$$

for all $\varphi \in \Phi$, C_1, C_2 real numbers. Then ξ, a measureable transformation of $(\Omega, \mathscr{A})$ into (K^*, K_r^*) is called a *random linear functional* on K.

Several of Ullrich's theorems and comments concerning random Schwartz distributions carry over for random functionals on K spaces without any change in the proofs. These include the following, the proofs being omitted:

U_1. If ξ is a random linear functional (rlf) on K, then there is a unique probability measure ν_0, defined on (K^*, K_r^*), given by $\mu\xi^{-1} = \nu_0$.

U_2. Let ξ be a mapping of Ω into K^*. Then ξ is a random linear functional on K if and only if for every $\varphi \in K$, $[\xi(w)](\varphi)$ is a complex random variable.

U_3. Let ξ_1, ξ_2 be random linear functionals on K, α a complex number. Define ζ_1, ζ_2 as follows:

$$\zeta_1(w) = \xi_1(w) + \xi_2(w), \qquad \zeta_2(w) = \alpha \cdot \xi_1;$$

then ζ_1, ζ_2 are also random linear functionals on K.

U_4. Let $\{\xi_m\}$ be a sequence of random linear functionals on K such that for all $w \in \Omega$, the sequence $\{\xi_m(w)\}$ converges in the topology of K^* to a mapping $\xi_0(w)$. Then ξ_0 is a random linear functional.

U_5. Let ξ be a random linear functional on K; then for every n-tuple, $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers, $D^\alpha \xi$ is a random linear functional, where

$$[D^\alpha \xi(w)](\varphi) = (-1)^{|\alpha|}[\xi(w)]\left(D^\alpha(\varphi)\right).$$

III. Some examples of random linear functionals.

A. Let Y be a real rector-valued random variable on $(\Omega, \mathscr{A})$; then from U_2 it is clear that $[\xi(w)](\varphi) = \varphi(Y(w))$ defines a random linear functional on K (or $\mathscr{K}$) and might be called a *random delta function*.

B. Let Y be a real vector-valued random variable on Ω such that the expected value, $E(\varphi(Y))$, exists for all φ in K. Suppose X is a real-valued random variable on $(\Omega, \mathscr{A})$ (dependent on Y). Then ξ_x defined up to an $P_{\mathscr{B}_X}$-equivalence by

$$[\xi_x(w)](\varphi) = E[\varphi(Y)|X = x](w)$$

(conditional expectation) is a random linear functional on K and might be considered a generalized time series. If X, Y are assumed to have joint and marginal density functions, $f(x, y)$ and $f(x)$ respectively, then

$$\text{(i)} \qquad \xi_x(\varphi) = \int_{R^n} \frac{f(x, y)}{f(x)} \varphi(y)\, dy$$

but $\xi_x(w) \in K^*$ implies that there exist functions $\{g_\alpha^x\}$, bounded measureable, such that

$$\text{(ii)} \qquad [\xi_x(w)](\varphi) = \sum_{|\alpha| \leqslant p} \int_{R^n} M_p D^\alpha \varphi(y) g_\alpha^x(y)\, dy,$$

where we assume condition (N) is satisfied (see [2], p. 113).

Equating the two representations, (i) and (ii) and assuming $f(x, y)$ as the unknown function, "testing" with functions in K could provide a method for obtaining approximate solutions.

C. Let $\{h_p(x, w)\}$ be complex-valued functions defined on $R^n \times \Omega$, jointly continuous in the first variable and measureable in the second. Further suppose that for each p and each $\varphi \in K$

$$X_p = \int_{R^n} M_p(x) h_p(x, w) \varphi(x) dx$$

converge absolutely. Then ξ defined by

$$[\xi(w)](\varphi) = \sum_{p=1}^{g} X_p(w)$$

is a random linear functional on K.

D. Let G be a measureable subset of R^n with positive measure. Construct $L_r^* = (L_r^1(G))^*$ as in the beginning of this section.

Proposition 1. *Let ξ be random linear functional on $L^1(G)$; then there exists* $g: \Omega \times G \to C$ such that

(i) *$g(\cdot, t)$ is measureable for all $t \in G$;*

(ii) *$g(w, \cdot)$ is essentially bounded on G (with respect to Lebesgue measure) for all $w \in \Omega$;*

(iii) *for all $w \in \Omega$, $\varphi \in L^1(G)$*

$$[\xi(w)](\varphi) = \int_G \varphi(t) g(w, t) dt.$$

Proof. Since $\xi(w) \in (L^1(G))^*$ there exists $h(w, \cdot) \in L^\infty(G)$ such that

$$[\xi(w)](\varphi) = \int_G \varphi(t) h(w, t) dt$$

for all $\varphi \in L^1(G)$. Extend h to all of R^n by setting $h(w, t) = 0$ for $t \notin G$, all $w \in \Omega$.

For arbitrary open sets in R^n, define

$$\mu_w(E) = \int_E h(w, t) dt = \int_G \varphi_E(t) h(w, t) dt = [\xi(w)](\varphi_E),$$

where φ_E is the characteristic function of E. By Theorem 8.6, [4], p. 154, for any substantial family of open sets, μ_w is differentiable a. e. $[m]$, $D\mu_w(x_0) = h(w, x_0)$ a. e. $[m]$ and hence $h(w, \cdot)$ is in $L^1(G)$. Since $w \to \mu_w(E)$ is measureable so is $w \to D\mu_w(x)$ for x in G. Let $g(w, x) = D\mu_w(x)$. Hence

$$[\xi(w)](\varphi) = \int_G \varphi(t) g(w, t) dt \quad \text{for all } \varphi \in L^1(G).$$

Since $h(w, \cdot)$ is essentially bounded on G so is $g(w, \cdot)$.

Although this example does not involve $K\{M_p\}$ spaces and in particular would not obtain a "differentiable" random linear functional, if G were assumed compact, then the Schwartz space $\mathcal{D}_G$ would be a subspace and the representation would still apply by first extending functionals to all of $L^1(G)$. This proposition also serves to illustrate the generality of Ullrich's definition of random linear functionals and also is used in the proof of Theorems 1 and 2.

IV. Representation theorems for random linear functionals.

Lemma 2. *Suppose $\{M_p\}$ satisfies conditions* (M), (N) *and* (P), *ξ is an rlf on K. Let $\epsilon > 0$; then:*

(i) *there exists $M \in \mathcal{A}$ such that $\mu(M) \geqslant 1 - \varepsilon$,*

(ii) *there exists $r > 0$ such that for all $\varphi \in K$, $w \in M$*

$$|[\xi(w)](\varphi)| \leqslant r \sup_{|\alpha|<r} \int_F M_p(t) |D^\alpha \varphi(t)| dt = r\|\varphi\|_r'.$$

Proof. Since $\xi(w)$ is in K^*, for all $w \in \Omega$ there exists $P_w > 0$ and $S_w \geqslant 0$ such that $|[\xi(w)](\varphi)| \leqslant S_w \|\varphi\|_{P_w}'$ for all $\varphi \in K$ (see [2], p. 112). Without loss of generality, we may assume that $P_w \geqslant S_w$ for all w and hence

$$|[\xi(w)](\varphi)| \leqslant P_w \|\varphi\|_{P_w}'.$$

Note that

$$\Omega = \bigcup_{N=1}^{\infty} \bigcap_{\varphi \in K} \{w \mid |[\xi(w)](\varphi)| \leqslant N \|\varphi\|_N'\} = \bigcup_{N=1}^{\infty} A_N(\varphi)$$

and $A_N(\varphi) \subseteq A_{N+1}(\varphi)$. Since K is separable, there exists a countable dense subset, H, and hence

$$A_N = \bigcap_{\varphi \in K} A_N(\varphi) = \bigcap_{\varphi \in H} A_N(\varphi)$$

is measureable subset of Ω. But $\Omega = \bigcup_{N=1}^{\infty} A_N$ implies there exists $r > 0$ such that $\mu(A_r) \geqslant 1 - \varepsilon$. Set $M = A_r$ and by the construction of $A_r(\varphi)$ and A_r, (ii) follows.

Remark. This is the analogue of Lemma 4 in [6].

Theorem 1. *Let ξ be an rlf on $\mathcal{K}$, where conditions* (M), (N) *and* (P) *are assumed for all sequences $\{M_p^j\}$. For m a positive integer and $\varepsilon > 0$ there exists an integer r, $M \in \mathcal{A}$ and functions $\{f_\alpha\}$, $|\alpha| \leqslant r$, such that*

(i) $\mu(M) \geqslant 1 - \varepsilon$,

(ii) *for each $w \in \Omega$, $f_\alpha(w, \cdot)$ is essentially bounded on S and for $t \in S$, $f_\alpha(\cdot, t)$ is measureable* $(|\alpha| \leqslant r)$,

(iii) *for all* $w \in M$, $\varphi \in K^m$

$$[\xi(w)](\varphi) = \sum_{|\alpha|\leqslant r} \int_{F^r} M_r^m(t) f_\alpha(w, t) D^\alpha \varphi(t)\, dt,$$

i. e. $\xi(w) = \sum_{|\alpha|\leqslant r} (-1)^{|\alpha|} D^\alpha [M_r^m f_\alpha(w, \cdot)]$.

Proof. (a) Since $\mathscr{K} = \varinjlim_{m\to\infty} K^m$, $\xi(w) \in \mathscr{K}^*$, $\xi(w) \in (K^m)^*$.

By Lemma 2, there exists $r > 0$, M measureable such that (i) is true and

$$|[\xi(w)](\varphi)| \leqslant r \sup_{|\alpha|\leqslant r} \int_{F^r} M_r^m(t) |D^\alpha \varphi(t)|\, dt = r\|\varphi\|_r'.$$

Extend ξ by setting

$$[\xi^*(w)](\varphi) = \begin{cases} [\xi(w)](\varphi), & w \in M, \\ 0, & w \notin M. \end{cases}$$

Set $\Lambda = \{\varphi | \varphi \in K^m, \|\varphi\|_r' \leqslant 1\}$

$$s(w) = \sup_{\varphi\in\Lambda} |[\xi^*(w)](\varphi)| = \sup_{\varphi\in H\cap\Lambda} |[\xi^*(w)](\varphi)|,$$

where H is a countable dense subset in K^m (see condition (P)). Then $s(w)$ is measureable and

$$|[\xi^*(w)](\varphi)| \leqslant s(w) \|\varphi\|_r'.$$

(b) For each $\varphi \in K^m$, associate a vector $\psi = \{\psi_\alpha\}$, where $\psi_\alpha = M_r^m D^\alpha \varphi$, $|\alpha| \leqslant r$. The correspondence $\theta: \varphi \to \psi$ is one-to-one. Let Γ be the direct sum of γ copies of $L^1(F^r)$, where ν is the number of components in φ. Norm Γ by

$$\|(f_1, \ldots, f_\nu)\| = \sup_{1\leqslant j\leqslant \nu} \|f_j\|_1, \quad \|f_j\|_1 = \int_{F^r} |f_j(t)|\, dt$$

and let Δ be the image of K^m under the map θ. Construct $L(w, \cdot)$ on Δ by $L(w, \theta(\varphi)) = [\xi^*(w)](\varphi)$ and note $L(w, \cdot)$ is in Δ^* with

$$|L(w, \theta(\varphi))| \leqslant s(w) \|\theta(\varphi)\|.$$

Since Γ is separable and $\Delta \subseteq \Gamma$, $L(w, \cdot)$ has an extension $L^*(w, \cdot)$ defined on Γ with $|L^*(w, x)| \leqslant s(w)\|x\|$ for all $x \in \Gamma$. This follows from Theorem 2, [3]. By Proposition 1, for all $|\alpha| \leqslant r$, there exists $g_\alpha: \Omega F^r \to R$ such that

(i) $g_\alpha(w, \cdot)$ is essentially bounded in F^r,

(ii) $g_\alpha(\cdot, t)$ is measureable,

(iii) $L^*(w, x) = \sum_{|\alpha|\leqslant r} \int_{F^r} x_\alpha(t) g_\alpha(w, t)\, dt$.

Hence

$$L(w, \theta(\varphi)) = [\xi(w)](\varphi) = \sum_{|\alpha|<r} \int_{F^r} M_r^m(t) g_\alpha(w, t) D^\alpha \varphi(t)\, dt$$

for $w \in M$.

Under appropriate conditions on the space $\mathscr{K}$, we will obtain a representation theorem analogous to Theorem 1, [6].

LEMMA 3. *Let S be a Lebesgue measureable subset of R^n and $f: \Omega \times S \to R$ such that*

(i) *$f(\cdot, t)$ is measureable for $t \in S$,*

(ii) *$f(w, \cdot)$ is continuous for all $w \in \Omega$.*

If $C \subseteq S$, is compact, then $g(w) = \int_C f(w, t)\, dt$ is Lebesgue measureable.

Proof. Since $f(w, \cdot)$ is continuous, for fixed w, the integral exists and is finite because C is compact.

Let

$$g_n(w) = \sum_{i=1}^{M_n} f(w, t_i) \Delta t_i,$$

where the t_i are chosen such that $|g_n(w) - g(w)| < 1/2^n$. By the existence of the integral this can be done but M_n and $\{t_i\}$ might be dependent on w. However, by the uniform continuity of $f(w, \cdot)$ on C, the M_n and $\{t_i\}$ are independent of w. We note finally that g_n is measureable for each n and hence that g is measureable.

LEMMA 4. *Let $f: \Omega \times S \to R$, $S \subset R^n$ be sure that*

(i) *$f(\cdot, t)$ is measureable for all $t \in S$,*

(ii) *$f(w, \cdot)$ is essentially bounded and measureable for all $w \in \Omega$ (and hence locally integrable).*

If $C \subseteq S$, is compact, then $g(w) = \int_C f(w, t)\, dt$ is measureable.

Proof. Since $f(w, \cdot)$ is measureable and bounded, for each w, there exists a sequence of continuous functions $f_n(w, \cdot)$ converging a. e. to $f(w, \cdot)$, and

$$\|f_n(w, \cdot)\|_\infty \leqslant \|f(w, \cdot)\|_\infty.$$

By Lemma 3 $\int_C f_n(w, t)\, dt$ is measureable and by the dominated convergence theorem

$$\lim_{n\to\infty} \int_C f_n(w, t)\, dt = \int_C f(w, t)\, dt = g(w)$$

is measureable since C is compact (i. e. C has finite measure).

THEOREM 2. *Let ξ be an rlf on $\mathscr{K}$, where conditions* (M), (N) *and* (P) *are assumed for all sequences $\{M_p^j\}$. Assume further that polynomials are multipliers for each K^m. For m a positive integer and $\varepsilon > 0$ there exists an integer and r, $M \in \mathscr{A}$ and functions $\{h_\alpha\}$, $|\alpha| \leqslant r$, such that*

(i) $\mu(M) \geqslant 1-\varepsilon$,

(ii) *$h_\alpha(w, \cdot)$ is continuous on F^r for all $w \in \Omega$,*

(iii) *$h_\alpha(\cdot, t)$ is measureable for $t \in F^r$,*

(iv) *for all $w \in M$, $\varphi \in K^m$*

$$[\xi(w)](\varphi) = \sum_{|\alpha|\leqslant r+1} (-1)^{|\alpha-1|} \int_{F^r} h_\alpha(w, t) D^\alpha \varphi(t)\, dt.$$

Proof. By Theorem 1, there exist functions f_α, $|\alpha| \leqslant r$, such that

$$\xi(w) = \sum_{|\alpha|\leqslant r} (-1)^{|\alpha|} D^\alpha [M_r^m f_\alpha(w, \cdot)],$$

where for each w, $f_\alpha(w, \cdot)$ is essentially bounded. If Lemma 1 is applied to each $f_\alpha(w, \cdot)$, then

$$\xi(w) = \sum_{|\alpha|\leqslant r} (-1)^{|\alpha|} D^{\alpha+1} \Big(\int_0^x M_p f_\alpha(w, \cdot) \Big)$$

and each

$$h_\alpha(w, x) = \int_0^x M_p(t) f_\alpha(w, t)\, dt$$

is continuous in x and measureable in w by an application of Lemma 4.

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Holomorphy types on a Banach space

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Contents

In discussing tensor products, bilinear mappings and linear mappings on a Banach space it has been found useful to distinguish between various sorts of mappings such as the compact, nuclear, integral mappings etc. (cf. Treves [17] and Grothendieck [2]). Since n-homogeneous polynomials are nothing more than symmetric n-linear mappings and a holomorphic function on a Banach space can be looked upon as a sequence of homogeneous polynomials which satisfy certain conditions, it is not surprising that one can define various subspaces of the space of all holomorphic functions so that the resulting structure is enriched. Such is the case in Nachbin and Gupta [15], where Malgrange's approximation theorem is generalized from the finite to the infinite-dimensional case. To describe a theory for a large class of subspaces Nachbin [13] introduced the concept of holomorphy type.

Motivated by Nachbin and Gupta [15] and Nachbin [13] we describe and study in this work various topological vector spaces of holomorphic functions.

In Section 1 we recall the definition of holomorphy type and of the spaces $(\mathscr{H}_\theta(E), \mathscr{T}_\theta)$. We define α-holomorphy type and the corresponding topological vectors spaces $(H_\theta(E), T_\theta)$.